A note on Talagrand's variance bound in terms of influences

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Abstract

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables and f a function on $\{0,1\}^n$. In the well-known paper [19] Talagrand gave an upper bound for the variance of f in terms of the individual influences of the X_i's. This bound turned out to be very useful, for instance in percolation theory and related fields.

In many situations a similar bound was needed for random variables taking more than two values. Generalizations of this type have indeed been obtained in the literature (see e.g. [8]), but the proofs are quite different from that in [19]. This might raise the impression that Talagrand's original method is not sufficiently robust to obtain such generalizations.

However, our paper gives an almost self-contained proof of the above mentioned generalization, by modifying step-by-step Talagrand's original proof.

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1 Introduction and statement of results

1.1 Statement of the main results

Let $(\Omega, \mathcal{F}, \mu)$ be an arbitrary probability space. We denote its n-fold product by itself by $(\Omega^n, \mathcal{F}^n, \mu^n)$. Let $f : \Omega^n \to \mathbb{C}$ be a function with finite second moment, that is $\int_{\Omega^n} |f|^2 d\mu^n < \infty$. The influence of the ith variable on the function f is defined as

$$\Delta_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \int_\Omega f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)\, \mu(d\xi)$$

for $x = (x_1, \ldots, x_n) \in \Omega^n$ and $i = 1, \ldots, n$. We will use the notation $\|f\|_q$ for the L^q norm $q \in [1, \infty)$ of f, that is $\|f\|_q = \sqrt[q]{\int_{\Omega^n} |f|^q d\mu^n}$.

Using Jensen's inequality, Efron and Stein gave the following upper bound on the variance of f (see [11]):

$$Var(f) \leq \sum_{i=1}^n \|\Delta_i f\|_2^2 . \tag{1.1}$$

In some cases (1.1) has been improved. We write $\mathcal{P}(S)$ for the power set of a set S. For the case when Ω has two elements, say 0 and 1, and $\mu(\{1\}) = 1 - \mu(\{0\}) = p$, Talagrand showed the following result:

Theorem 1.1 (Theorem 1.5 of [19]). *There exists a universal constant K such that for every $p \in (0,1)$, $n \in \mathbb{N}$ and for every real valued function f on $(\{0,1\}^n, \mathcal{P}(\{0,1\}^n), \mu_p)$,*

$$Var(f) \leq K \log\left(\frac{2}{p(1-p)}\right) \sum_{i=1}^n \frac{\|\Delta_i f\|_2^2}{\log\left(e \|\Delta_i f\|_2 / \|\Delta_i f\|_1\right)}, \tag{1.2}$$

where μ_p is the product measure on $\{0,1\}^n$ with parameter p.

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Remark 1.2. An alternative proof of Theorem 1.1 for the case $p = 1/2$ can be found in [4].

Inequality (1.2) gives a bound on $Var(f)$ in terms of the influences. It is useful when the function f is complicated, but its influences are tractable. Such situations occur for example in percolation theory (see for example [4, 6, 20]). Further consequences of (1.2) include for example the widely used KKL lower bound for influences [15] and various so called sharp-threshold results e.g. [13].

In some cases, a generalization of Theorem 1.1 to the case $\{0, 1, \ldots, k\}^n$ with $k > 1$ is useful, for example in [7, 9]. However, up to our knowledge, no such generalization has been explicitly stated in the literature. The main goal of our paper is to present and prove an explicit generalization, Theorem 1.3 below. We have used this theorem and referred to it in [21].

Theorem 1.3. *There is a universal constant $K > 0$ such that for each finite set Ω each measure μ on Ω with $p_{min} = \min_{j\in\Omega} \mu(\{j\}) > 0$, and for all complex valued functions f on $(\Omega^n, \mathcal{P}(\Omega^n), \mu^n)$,*

$$Var(f) \le K \log(1/p_{min}) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e \|\Delta_i f\|_2 / \|\Delta_i f\|_1\right)}. \tag{1.3}$$

Remark 1.4. Inequality (1.3) is sharp up to a universal constant factor, which can easily be seen by taking the function $f(x) = 1$ if $x_i = \omega$ for all $i = 1, \ldots, n$ where ω is some element of Ω is such that $\mu(\{\omega\}) = p_{min}$, and $f(x) = 0$ otherwise.

Herein, we follow the line of argument of Talagrand [19] and modify his symmetrization procedure to deduce the result above. Given the paper of Talagrand [19], the proof is self contained apart from Lemma 1 of [9].

Cordero-Erausquin and Ledoux [8] in a recent preprint further generalized Theorem 1.3, however their approach is very different from the original proof of Talagrand. (One can deduce a result, equivalent up to a universal constant to Theorem 1.3, from Theorem 1 of [8], by combining it with Theorem A.1 of [10]. This results in a slightly more complicated proof.)

We finish this section by noting that the special case of Theorem 1.3, where μ^n is the uniform measure on Ω^n, has been proved in [9].

1.2 Background and further motivation for Theorem 1.3

Falik and Samarodnitsky [12] used logarithmic Sobolev inequalities to derive edge isoperimetric inequalities. Rossignol used this method to derive sharp threshold results [17, 18]. Furthermore, Benaïm and Rossignol [3] extended the results of [4] (where Talagrand's Theorem 1.1 above is applied to first-passage percolation), again with the use of logarithmic Sobolev inequalities. These similar applications suggest a deeper connection between logarithmic Sobolev inequalities and (1.2). Indeed, Bobkov and Houdré in [5], proved that a version of (1.2) actually implies a logarithmic Sobolev inequality in a continuous set-up. Moreover, Cordero-Erausquin and Ledoux in [8] showed the same implication under different assumptions.

Another motivation for Theorem 1.3 is to point out the following mistake in the literature. We borrow the notation of [14]. For any $x \in \Omega^n$ and $i = 1, \ldots, n$, we define

$$s_i(x) = \{y \in \Omega^n \mid y_j = x_j \text{ for all } j \ne i\}.$$

For $i = 1, \ldots, n$, let $I_f(i)$ denote the probability of the event that the value of f does depend on the ith coordinate, that is

$$I_f(i) = \mu^n\left(\{x \in \Omega^n : f \text{ is non-constant on } s_i(x)\}\right).$$

The following claim, which is related to our Theorem 1.3 was stated as Theorem 3.3 in [14]. However, as we will show, this claim is incorrect.

For any probability space $(\Omega, \mathcal{F}, \mu)$, and positive integer n, for any square integrable function $f : (\Omega^n, \mathcal{F}^n, \mu^n) \to \mathbb{R}$, we have

$$Var(f) \le 10 \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log(1/I_f(i))}. \tag{1.4}$$

One can easily see, that the following is a counterexample for this claim. Let k be an arbitrary positive integer. Take $n = 2$ and consider the case where $\Omega = [0, 1]$ and μ is the uniform measure. Take the function f (similar to the function in Remark 1.4) defined as $f(x_1, x_2) = 1$ if $0 \leq x_1, x_2 \leq 1/k$ and 0 otherwise. Substituting to (1.4) and choosing k large enough, we get a contradiction.

Note that we can easily salvage (1.4) under the conditions of Theorem 1.3. If in equation (1.4) we replace the constant 10 for $K \log(1/p_{min})$, we get a valid statement, since we it follows from (1.3) by applying second moment method in the denominator.

Most of the aforementioned applications of the inequality (1.2) are concerned with the special case where $f = \mathbf{1}_A$, that is f is the indicator function of some event $A \subseteq \Omega^n$. We warn the reader about the slight inconsistency of the literature: $I_A(i)$ is called the influence of the ith variable on the event A, instead of some $L^p, p \geq 1$ norm of $\Delta_i f = \Delta_i \mathbf{1}_A$, which is the usual influence for arbitrary functions. For comparison of different definitions of influence, see e.g. [16].

Note that

$$\|\Delta_i \mathbf{1}_A\|_2^2 = \|\Delta_i \mathbf{1}_A\|_1 \leq p_{med} \mu^n (A_i), \tag{1.5}$$

where $p_{med} = \max \left\{ \mu(B) | B \subset \Omega, \mu(B) \leq \frac{1}{2} \right\}$. Using this we can deduce the following generalization of Corollary 1.2 of [19].

Corollary 1.5. *There is a universal constant $C > 0$ such that for each finite set Ω and each measure μ on Ω and for sets $A \subseteq \Omega^n$,*

$$\sum_{i=1}^{n} I_A(i) \geq C \frac{\log (1/ \max_i I_A(i))}{p_{med} \log (1/p_{min})} \mu^n(A) \left(1 - \mu^n (A)\right). \tag{1.6}$$

Using the corollary above, one can easily deduce the sharp threshold results of [7].

We finish this introduction with some remarks on the proof of Theorem 1.3. The proof of Theorem 1.5 of [19] uses a hypercontractive result (Bonami-Beckner inequality, see [2]) followed by a subtle symmetrization procedure (see Step 2 and 3 of the proof of Lemma 2.1 in [19]). In the proof of our more general Theorem 1.3 above, we use a consequence of the extended Bonami-Beckner inequality (for an extension of the Bonami-Beckner inequality see Claim 3.1 in [1]) from [9] and then modify Talagrand's symmetrization procedure. This generalization of Talagrand's symmetrization argument, which covers Sections 2.2 and 2.3 is the main part of our proof.

2 Proof of Theorem 1.3

Without loss of generality, we assume that $\Omega = \mathbb{Z}_k$ (the integers modulo k) for some $k \in \mathbb{N}$.

Let η be an arbitrary measure on $\mathbb{Z}_k^n$. For each η, we will write $L_\eta(\mathbb{Z}_k^n)$ for the (Hilbert) space of complex valued functions on $\mathbb{Z}_k^n$, with the inner product

$$\langle f, g \rangle_\eta = \int_{\mathbb{Z}_k^n} f \overline{g} d\eta \text{ for } f, g \in L_\eta(\mathbb{Z}_k^n).$$

We will write $\|f\|_{L^q(\eta)}$ for the q-norm, $q \in [1, \infty)$, of a function $f : \mathbb{Z}_k^n \to \mathbb{C}$ with respect to the measure η, that is

$$\|f\|_{L^q(\eta)} = \left(\int |f|^q \, d\eta \right)^{1/q}.$$

When it is clear from the context which measure we are working with, we will simply write $\|f\|_q$.

2.1 A hypercontractive inequality

Let ν^n denote the uniform measure on $\mathbb{Z}_k^n$. Define the "scalar product" on $\mathbb{Z}_k^n$ by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \text{ for } x, y \in \mathbb{Z}_k^n.$$

Let $\varepsilon = e^{2\pi i/k}$. For every $y \in \mathbb{Z}_k^n$, define the functions

$$w_y(x) = \varepsilon^{\langle x, y \rangle} \text{ for } x \in \mathbb{Z}_k^n.$$

It is easy to check the following lemma.

Lemma 2.1. $\{w_y\}_{y \in \mathbb{Z}_k^n}$ *form an orthonormal basis in* $L_{\nu^n}(\mathbb{Z}_k^n)$.

Let us denote the number of non-zero coordinates of $\xi \in \mathbb{Z}_k^n$ by $[\xi]$. We will use the following hypercontractive inequality:

Lemma 2.2. *(Lemma 1 of [9]) There are positive constants* C, γ *such such that for any* $k, n \in \mathbb{N}$, $m \in \{0, 1, \ldots, n\}$ *and complex numbers* a_y, *for* $y \in \mathbb{Z}_k^n$, *we have*

$$\left\| \sum_{[y]=m} a_y w_y \right\|_{L^4(\nu^n)} \leq (Ck^\gamma)^m \left(\sum_{[y]=m} |a_y|^2 \right)^{1/2}. \tag{2.1}$$

Remark 2.3. The proof (in [9]) of Lemma 2.2 is based on Claim 3.1 of [1]. Claim 3.1 of [1] is a generalization of the so called Bonami-Beckner inequality (see Lemma 1 of [2]). That inequality played an important role in [19] in the original proof of Theorem 1.1.

2.2 Finding a suitable basis

We assume that $\mu(\{j\}) > 0$ for all $j \in \mathbb{Z}_k$. Let $L_\mu(\mathbb{Z}_k)$ be the Hilbert space of functions from $\mathbb{Z}_k$ to $\mathbb{C}$, with the inner product

$$\langle a, b \rangle_\mu = \sum_{j \in \mathbb{Z}_k} a(j) \overline{b(j)} \mu(\{j\}) \text{ for } a, b \in L_\mu(\mathbb{Z}_k).$$

Let $c_0 \in L_\mu(\mathbb{Z}_k)$ be the constant 1 function. By Gram-Schmidt orthogonalization, there exist functions $c_l \in L_\mu(\mathbb{Z}_k)$ for $l \in \mathbb{Z}_k \setminus \{0\}$, such that c_j, $j \in \mathbb{Z}_k$ form an orthonormal basis in $L_\mu(\mathbb{Z}_k)$.

Using the functions c_j, $j \in \mathbb{Z}_k$ we define an orthonormal basis in $L_{\mu^n}(\mathbb{Z}_k^n)$ analogous to the basis w_y, $y \in \mathbb{Z}_k^n$. It is easy to check the following lemma.

Lemma 2.4. *The functions* u_y, *for* $y \in \mathbb{Z}_k^n$, *defined by*

$$u_y(x) = \prod_{i=1}^{n} c_{y_i}(x_i) \text{ for } x \in \mathbb{Z}_k^n, \tag{2.2}$$

form an orthonormal basis in $L_\mu(\mathbb{Z}_k^n)$.

2.3 Extension of Lemma 2.2

The key ingredient in the proof of Theorem 1.3 is the following generalization of Lemma 2.2. It can also be seen as an extension of Lemma 2.1 of [19]. One could also use Theorem 2.2 of [22], however the proof of that theorem is more complicated.

Lemma 2.5. *With the constants of Lemma 2.2, we have for every $k,n\in\mathbb{N}$, $m\in\{0,1,\ldots,n\}$ and complex numbers a_y, $y\in\mathbb{Z}_k^n$,*

$$\left\|\sum_{[y]=m}a_yu_y\right\|_{L^4(\mu^n)}\leq\left(C\theta k^{\gamma}\right)^m\left(\sum_{[y]=m}\left|a_y\right|^2\right)^{1/2}\tag{2.3}$$

holds, where $\theta=k\max_{i,j}\left|c_i\left(j\right)\right|$.

Proof. The proof generalizes the symmetrization technique of the proof of Lemma 2.1 of [19]. Recall the definitions of ε and w_y for $y\in\mathbb{Z}_k^n$ of Section 2.1. Let n,k,m and the numbers a_y $y\in\mathbb{Z}_k^n$ as in the statement of Lemma 2.2.

Step 1 Define the product space $G=\left(\mathbb{Z}_k^n\right)^k$ with the probability measure $\mu_k^n=\bigotimes_{i=1}^k\mu$. For $y,z\in\mathbb{Z}_k^n$ define the functions $g_y,g_{y,z}$ on G as follows. For $X=\left(X^0,\ldots,X^{k-1}\right)\in\left(\mathbb{Z}_k^n\right)^k$ and $z\in\mathbb{Z}_k^n$, let

$$g_y\left(X\right)=\prod_{1\leq i\leq n,\,y_i\neq0}\sum_{l=0}^{k-1}c_{y_i}(X_i^l)\varepsilon^{ly_i},\tag{2.4}$$

$$g_{y,z}\left(X\right)=\prod_{1\leq i\leq n,\,y_i\neq0}\varepsilon^{z_iy_i}\sum_{l=0}^{k-1}c_{y_i}(X_i^l)\varepsilon^{ly_i}=g_y\left(X\right)w_y(z).\tag{2.5}$$

Recall that ν is the uniform measure on $\mathbb{Z}_k^n$, and define the set $H=G\times\mathbb{Z}_k^n$ and the product measure $\kappa=\mu_k\otimes\nu$ on H. We also define, for $y\in\mathbb{Z}_k^n$ the functions h_y on H by $h_y\left(X,z\right)=g_{y,z}\left(X\right)=g_y\left(X\right)w_y(z)$.

Step 2 For X as before and for $z\in\mathbb{Z}_k^n$ define X_z as

$$\left(X_z\right)_i^l=X_i^{l+z_i\mod k}.$$

Then

$$\begin{aligned}g_{y,z}\left(X_z\right)&=\prod_{1\leq i\leq n,\,y_i\neq0}\sum_{l=0}^{k-1}c_{y_i}(X_i^{l+z_i\mod k})\varepsilon^{(l+z_i)y_i}\\&=\prod_{1\leq i\leq n,\,y_i\neq0}\sum_{l=0}^{k-1}c_{y_i}(X_i^l)\varepsilon^{ly_i}=g_y\left(X\right).\end{aligned}$$

Hence for each fixed $z\in\mathbb{Z}_k^n$, we have

$$\left\|\sum_{[y]=m}a_yg_y\right\|_{L^4\left(\mu_k^n\right)}=\left\|\sum_{[y]=m}a_yg_{y,z}\right\|_{L^4\left(\mu_k^n\right)}.\tag{2.6}$$

Integrating over the variable z with respect to ν^n, Fubini's theorem gives that

$$\left\|\sum_{[y]=m}a_yg_y\right\|_{L^4\left(\mu_k^n\right)}=\left\|\sum_{[y]=m}a_yh_y\right\|_{L^4(\kappa)}.\tag{2.7}$$

Step 3 For fixed X, use Lemma 2.2 for the numbers $a_yg_y\left(X\right)$, and get

$$\int\left|\sum_{[y]=m}a_yg_y\left(X\right)w_y\left(z\right)\right|^4d\nu^n(z)\leq\left(Ck^{\gamma}\right)^{4m}\left(\sum_{[y]=m}\left|a_yg_y\left(X\right)\right|^2\right)^2.\tag{2.8}$$

Since $\theta = k \max_{i,j} |c_i(j)|$, we have that $|g_y(X)| \le \theta^m$, which together with (2.8) gives

$$\int \left| \sum_{[y]=m} a_y g_y(X) w_y(z) \right|^4 d\nu^n(z) \le (C\theta k^\gamma)^{4m} \left(\sum_{[y]=m} |a_y|^2 \right)^2 .$$

Integrating with respect to $d\mu_k(X)$ and taking the 4th root gives

$$\left\| \sum_{[y]=m} a_y h_y \right\|_{L^4(\kappa)} \le (C\theta k^\gamma)^m \left(\sum_{[y]=m} |a_y|^2 \right)^{1/2} . \tag{2.9}$$

By (2.9) and (2.7) we only have to show that

$$\left\| \sum_{[y]=m} a_y u_y \right\|_{L^4(\mu^n)} \le \left\| \sum_{[y]=m} a_y g_y \right\|_{L^4\left(\mu_k^n\right)} . \tag{2.10}$$

Step 4 Now we prove an alternative form of the function g_y. Recall the definition (2.4) of g_y. Expand the product, and get

$$\begin{aligned} g_y(X) &= \prod_{1 \le i \le n,\, y_i \ne 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l) \varepsilon^{l y_i} \\ &= \sum_{\alpha:(*)} \prod_{1 \le i \le n,\, y_i \ne 0} c_{y_i}(X_i^{\alpha(i)}) \varepsilon^{\alpha(i) y_i}, \end{aligned} \tag{2.11}$$

where $(*)$ denotes the sum over all functions $\alpha : \{i \,|\, y_i \ne 0\} \to \mathbb{Z}_k$.

We will use the following trivial observation:

Observation: $c_{y_i}(X_i^l)\varepsilon^{l y_i} = 1$ whenever $y_i = 0$.

With the Observation we rewrite (2.11) as follows.

$$\begin{aligned} g_y(X) &= \sum_{\alpha \in \mathcal{A}_y} \prod_{i=1}^{n} c_{y_i}(X_i^{\alpha(i)}) \varepsilon^{\alpha(i) y_i} \\ &= \sum_{\alpha \in \mathcal{A}_y} \prod_{t \in \mathbb{Z}_k} \prod_{1 \le i \le n,\, \alpha(i)=t} c_{y_i}(X_i^t) \varepsilon^{t y_i}, \end{aligned} \tag{2.12}$$

where $\mathcal{A}_y$ is the set of functions $\alpha : \{1, 2, \ldots, n\} \to \mathbb{Z}_k$ with the property that $\alpha(i) = 0$ if $y_i = 0$. For a function $\alpha \in \mathcal{A}_y$ we can define the vectors $v^t = v^t(\alpha) \in \mathbb{Z}_k^n$ for $t \in \mathbb{Z}_k$ by

$$v_i^t = v_i^t(\alpha) = \begin{cases} y_i & \text{if } \alpha(i) = t \\ 0 & \text{otherwise.} \end{cases}$$

The map $\alpha \mapsto (v^t(\alpha))_{t \in \mathbb{Z}_k}$ is one-to-one, furthermore the image of $\mathcal{A}_y$ under this map is

$$\mathcal{V}_y = \left\{ v = \left(v^t\right)_{t \in \mathbb{Z}_k} \,\middle|\, \sum_{t \in \mathbb{Z}_k} v^t = y, \text{ and } \forall i\, v_i^t \ne 0 \text{ for at most one } t \in \mathbb{Z}_k \right\}.$$

Using the properties of the map $\alpha \mapsto (v^t(\alpha))_{t\in\mathbb{Z}_k}$ together with the Observation and the definition of u, we can conclude from (2.12) that

$$\begin{aligned} g_y(X) &= \sum_{v\in\mathcal{V}_y} \prod_{t\in\mathbb{Z}_k} \prod_{i=1}^{n} c_{v_i^t}(X_i^t)\varepsilon^{tv_i^t} \\ &= \sum_{v\in\mathcal{V}_y} \prod_{t\in\mathbb{Z}_k} u_{v^t}(X^t)\varepsilon^{t\langle v^t,\mathbf{1}\rangle} \end{aligned} \tag{2.13}$$

where $\mathbf{1}$ is vector in $\mathbb{Z}_k^n$ with all coordinates equal to 1.

Step 5 Now we prove (2.10). Jensen's inequality gives that

$$\begin{aligned} \int \left| \sum_{[y]=m} a_y g_y(X) \right|^4 d\mu_k^n(X) &\geq \int \left| \int \sum_{[y]=m} a_y g_y(X)\, d\mu_{k-1}^n\left(X^1,\ldots,X^{k-1}\right) \right|^4 d\mu^n\left(X^0\right) \\ &= \int \left| \sum_{[y]=m} a_y \int g_y(X)\, d\mu_{k-1}^n\left(X^1,\ldots,X^{k-1}\right) \right|^4 d\mu^n\left(X^0\right). \end{aligned} \tag{2.14}$$

By (2.13), the inner integral of the left hand side of (2.14) is

$$\int g_y(X)\, d\mu_{k-1}^n\left(X^1,\ldots,X^{k-1}\right) = \int \sum_{v\in\mathcal{V}_y} \prod_{t\in\mathbb{Z}_k} u_{v^t}(X^t)\varepsilon^{t\langle v^t,\mathbf{1}\rangle} d\mu_{k-1}^n\left(X^1,\ldots,X^{k-1}\right) \tag{2.15}$$

$$= \sum_{v\in\mathcal{V}_y} \left(\prod_{t\in\mathbb{Z}_k} \varepsilon^{t\langle v^t,\mathbf{1}\rangle} \right) u_{v^0}(X^0) \prod_{l=1}^{k-1} \int u_{v^l}(X^l) d\mu^n\left(X^l\right). \tag{2.16}$$

Since u_0 is the constant 1 function on $\mathbb{Z}_k^n$, and by Lemma 2.4 (u_w, $w\in\mathbb{Z}_k^n$) is an orthonormal basis of $L_\mu\left(\mathbb{Z}_k^n\right)$, we have

$$\int u_w d\mu^n = \int u_w u_0 d\mu^n = \begin{cases} 1 & \text{if } w=0 \\ 0 & \text{otherwise.} \end{cases}$$

By this and the definition of $\mathcal{V}_y$ we conclude from (2.16) that

$$\int g_y(X)\, d\mu_{k-1}^n\left(X^1,\ldots,X^{k-1}\right) = \sum_{v\in\mathcal{V}_y,\, v^1=\ldots=v^{k-1}=0} \left(\prod_{t\in\mathbb{Z}_k} \varepsilon^{t\langle v^t,\mathbf{1}\rangle} \right) u_{v^0}(X^0) = u_y(X^0). \tag{2.17}$$

(2.17) together with (2.14) gives that

$$\int \left| \sum_{[y]=m} a_y g_y(X) \right|^4 d\mu_k^n(X) \geq \int \left| \sum_{[y]=m} a_y u_y(X^0) \right|^4 d\mu^n\left(X^0\right),$$

from which by taking the 4th root, we get (2.10). This completes the proof of Lemma (2.5). $\square$

From Lemma (2.5) and duality, we conclude the following lemma.

Lemma 2.6. *With the constants of Lemma 2.2, for any function $g\in L_\mu\left(\mathbb{Z}_k^n\right)$ we have*

$$\sum_{[y]=l} |\hat{g}(y)|^2 \leq \left(C\theta k^\gamma\right)^{2l} \|g\|_{L^{4/3}(\mu)}^2.$$

2.4 Completion of the proof of Theorem 1.3

Notice that

$$\begin{aligned}\int_\Omega u_y\left(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n\right)\mu\left(d\xi\right) &= \sum_{j\in\mathbb{Z}_k} c_{y_i}(j)\mu\left(\{j\}\right)\prod_{1\le l\le n\, l\ne i} c_{y_l}\left(x_l\right)\\ &= \left\langle c_{y_i},c_0\right\rangle_\mu \prod_{1\le l\le n\, l\ne i} c_{y_l}\left(x_l\right)\\ &= \begin{cases} u_y\left(x\right) & \text{if } y_i=0\\ 0 & \text{if } y_i\ne 0.\end{cases}\end{aligned}$$

Hence

$$\int_\Omega f\left(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n\right)\mu\left(d\xi\right) = \sum_{y\in\mathbb{Z}_k^n,\, y_i=0}\hat{f}(y)u_y$$

where $f=\sum_y \hat{f}(y)u_y$, i.e $\hat{f}(y)=\left\langle f,u_y\right\rangle_\mu$.
By the definition of $\Delta_i f$ we have

$$\Delta_i f = \sum_{y\in\mathbb{Z}_k^n,\, y_i\ne 0}\hat{f}(y)u_y. \tag{2.18}$$

Recall that $[y]$ was the number of non-zero coordinates of a vector $y\in\mathbb{Z}_k$. Define $M(g)$ by

$$M\left(g\right)^2=\sum_{y\in\mathbb{Z}_k^n,\, y\ne 0}\frac{\hat{g}\left(y\right)^2}{[y]}\text{ for } g\in L_\mu\left(\mathbb{Z}_k^n\right).$$

Take a function $f\in L_\mu\left(\mathbb{Z}_k^n\right)$ with $\int f d\mu=0$ (which is equivalent to $\hat{f}(0)=0$). Then Parseval's formula and (2.18) gives that

$$\|f\|^2_{L^2(\mu^n)}=\sum_{y\ne 0}\hat{f}(y)^2=\sum_{i=1}^n M(\Delta_i f)^2. \tag{2.19}$$

Since $1=\sum_{j=0}^{k-1}\left|c_i\left(j\right)\right|^2 p_j$, we can conclude that $\theta\le k/\min_j\sqrt{p_j}$. Hence Theorem 1.3 follows from the Proposition 2.7 bellow and (2.19).

Proposition 2.7. *There is a positive constant K, such that if $\int g d\mu=0$, we have*

$$M(g)^2\le K\log\left(C\theta k^\gamma\right)\frac{\|g\|_2^2}{\log\left(e\|g\|_2/\|g\|_1\right)},$$

where $\theta=k\max_{i=1,\ldots,n\, j\in\mathbb{Z}_k}\left|c_i\left(j\right)\right|$, and the constants C,γ are the same as in Lemma 2.2.

Proof. The proof of Proposition (2.7) is the same as the proof of Proposition 2.3 in [19] with the following modifications. Take $q=4$ instead of $q=3$, and use Lemma 2.6 instead of Proposition 2.2 of [19]. The only difference will be in the constants. First we get the term $2\log\left(C\theta k^\gamma\right)$ in stead of $\log\left(2\theta^2\right)$. Furthermore we have to replace the estimate

$$\frac{\|g\|_2}{\|g\|_1}\le\left(\frac{\|g\|_2}{\|g\|_{3/2}}\right)^3$$

by

$$\frac{\|g\|_2}{\|g\|_1}\le\left(\frac{\|g\|_2}{\|g\|_{4/3}}\right)^2,$$

which is a consequence of the Cauchy-Schwartz inequality. This substitution only affects the constant K.
This completes the proof of Proposition (2.7) and the proof of Theorem 1.3. $\square$

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